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# Generalized long-range ferromagnetic Ising spin models

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**Abstract.** A class of simply solvable long-range ferromagnetic models is studied in terms of the eigenvalues and eigenvectors of the interaction matrix. The validity of this approach was first systematically studied by Canning [1]. The present paper is a companion paper to this one. The generalized ferromagnetic models studied in this paper are a ferromagnetic equivalent of Hopfield neural networks and site-disorder spin glass models, although the interactions of the examples studied in this paper are chosen in a deterministic way. Our ferromagnetic models, in the same way as the separable disordered models, are described by Curie–Weiss mean-field equations of the form $\langle S_i\rangle = \tanh\beta(\sum_j J_{ij}\langle S_j\rangle)$, and have a free energy surface with many minima (but finite in number) separated by infinite energy barriers. They have stable states (in the sense that they have an infinite lifetime in the thermodynamic limit) which are non-ferromagnetic, although the ferromagnetic stable states always have the lowest free energy.

## 1. Introduction

In recent years there has been a resurgence in the study of long-range Ising spin models. This is partly due to the continued interest in long-range spin glass models [2] but mainly due to the recent explosion of interest in Ising spin neural networks [3]. In this paper we will present a class of long-range Ising spin models closely related to these models but very much simpler in the sense that they do not have a diverging (in the thermodynamic limit) number of metastable states at low temperatures. Our models are closely related to the separable site-disorder spin glasses [4–8] and their earlier precursors [9, 10], and to the Hopfield neural network storing a finite number of patterns [11]. The models studied in this paper can be defined in a similar way to these models by constructing their interactions from deterministic, rather than random, quenched variables which sit on each site (see the section in this paper on two weakly coupled ferromagnetic systems for an example). Generalized ferromagnetic models with random couplings can also be defined, such as the random bond model studied in [1].

The definition of our class of models (generalized long-range ferromagnetic models) is that the interactions between Ising spins satisfy $J_{ij} \geqslant 0$, and the number of interactions per spin scales in some way with the system size, such that it diverges in the thermodynamic limit. The long-range Ising ferromagnetic model ($J_{ij} = J/N$, for all $i, j$) studied by Kac [12] is the simplest example of a long-range ferromagnet. The systems we wish to study in this paper generalize this model and typically have fewer interactions per spin, or the interactions are not all of the same magnitude. Before going on to study a few examples of this class of system we will briefly review the saddle-point mean-field theory appropriate for describing these types of models.

In reference [1] the saddle-point mean-field theory for long-range Ising spin models was presented in terms of the eigenvalues and eigenvectors of the interaction matrix. This paper showed how all long-range Ising spin models defined by a Hamiltonian of the form

$$H = -\frac{1}{2}\sum_{i,j} S_i J_{ij} S_j \tag{1}$$

(the sum is over all $i$ and $j$) could be divided into two classes: those where the rank of the interaction matrix $\mathbf{R}(J)$ remains finite in the thermodynamic limit, and those where $\mathbf{R}(J)$ diverges in the thermodynamic limit. In the former case the finiteness of $\mathbf{R}(J)$, plus a few other weaker conditions, were shown to be sufficient conditions for the system to be described by Curie–Weiss type mean-field equations of the form $\langle S_i\rangle = \tanh\beta(\sum_j J_{ij}\langle S_j\rangle)$. The different stable states of the system can then be characterized by the finite set of order parameters associated with the non-zero eigenvalues which are given by

$$m_q = \frac{1}{\sqrt{N}}\sum_i V_q^i \tanh\left[\beta\sqrt{N}\sum_k m_k\lambda_k V_k^i\right]. \tag{2}$$

$\lambda_k$, $(k = 1,\ldots,s)$ are the finite set of $s$ non-zero eigenvalues of $J$ with corresponding normalised eigenvectors $V_k$ therefore the sum $\sum_k$ only goes from 1 to $s$. The physical interpretation of these order parameters is given by

$$m_q = \frac{1}{\sqrt{N}}\sum_i V_q^i\langle S_i\rangle \tag{3}$$

and the free energy per site is

$$f = \frac{1}{2}\beta\sum_q \lambda_q m_q^2 - \frac{1}{N}\left[\prod_i 2\cosh\beta\sqrt{N}\sum_q m_q\lambda_q V_q^i\right]. \tag{4}$$

Examples of systems which have $\mathbf{R}(J)$ finite in the thermodynamic limit and are described by these mean-field equations are: separable site-disorder spin glass models [5–8] the Hopfield neural network storing a finite number of patterns [11]; and the generalized ferromagnetic models studied in this paper. Models falling into the other class ($\mathbf{R}(J)$ divergent in the thermodynamic limit) are typically studied using other techniques such as the replica method in the case of the SK [13] spin glass [2]. We will now look at a few specific examples of long-range generalized ferromagnetic models.

## 2. Examples of generalized ferromagnet models

### 2.1. Two weakly coupled ferromagnetic systems

This is the simplest model which exhibits the properties we wish to illustrate, so we will start this section by studying this model in detail. The model consists of two subsystems of $\frac{1}{2}N$ spins in which all the spins in each subsystem interact with each other, and we then couple these two subsystems together. We will study two different ways of coupling the two subsystems. The simplest, which we shall study first, is to couple all the spins between different subsystems together with a coupling which is weaker than the interaction between spins in the same subsystem. Secondly, we will study the case where we couple together only a percentage of the spins between the two subsystems with interactions having the same strength as those between spins in the same subsystem.

We will start by studying the problem of two decoupled subsystems of a $\frac{1}{2}N$ spins considered as a whole system of $N$ spins. The results from this calculation will be useful later when we couple the two subsystems together. The interaction matrix for systems of this type can be written in many ways, the most natural of which would be a four block structure but the problem of finding the eigenvalues and eigenvectors is mathematically more convenient if we formulate it in a translationally invariant way. We also wish to present this model as a case study representative of all translationally invariant long-range generalized ferromagnetic models. The interaction architecture of our model can be formulated in a one dimensional way, which means the eigenvalues and eigenvectors can be expressed as

$$\begin{aligned} \lambda_q &= \sum_r J(r) \exp(2\pi i r q) \\ V_q^j &= \frac{1}{\sqrt{N}} \exp(2\pi i q j) \end{aligned} \tag{5}$$

where $r = 0, 1, \ldots, N-1$ and $q = 0, 1/N, \ldots, (N-1)/N$ are the reciprocal lattice vectors, and $J_{ij} = J(r)$ where $r = |i-j|$. The eigenvectors of a matrix with translational invariance do not depend on the specific choice of $J(r)$, so the order parameters describing the stable states of the system will not change as we couple the two subsystems together.

Two decoupled spin subsystems of $\frac{1}{2}N$ spins can be represented by an interaction matrix defined by

$$J(r) = \frac{1}{N} (1\ \ 0\ \ 1\ \ 0 \ldots 1\ \ 0) \tag{6}$$

where for simplicity we have chosen the interaction strength to be $1/N$. The diagonal terms have been chosen to be $J_{ii} = J(0) = 1/N$. With this choice for the interactions the matrix $\mathbf{J}$ has only two non-zero eigenvalues with associated eigenvectors which are

$$\begin{aligned} \lambda_0 &= \tfrac{1}{2} \qquad V_0 = \frac{1}{\sqrt{N}} (1\ \ 1\ \ 1\ \ 1\ \ldots\ 1\ \ 1) \\ \lambda_{\frac{1}{2}} &= \tfrac{1}{2} \qquad V_{\frac{1}{2}} = \frac{1}{\sqrt{N}} (1\ -1\ \ 1\ -1\ \ldots\ 1\ -1). \end{aligned} \tag{7}$$

Thus, the interaction matrix $\mathbf{R}(J)$ has a finite rank and the eigenvalues and eigenvectors satisfy the weaker conditions (conditions 2 and 3 in reference [1]) for the Curie–Weiss mean-field theory to be valid. The system is described by two mean-field order parameters, which we shall denote $m_0$ and $m_{\frac{1}{2}}$, associated with the eigenvectors $V_0$ and $V_{\frac{1}{2}}$ and defined by equation (2). The only minima in the free energy function $f(m_0, m_{\frac{1}{2}}, T)$ (see equation (4)) are given by

$$\begin{aligned} & m_0 = \tanh \lambda_0 \beta m_0 \qquad m_{\frac{1}{2}} = 0 \\ & \text{or} \\ & m_{\frac{1}{2}} = \tanh \lambda_{\frac{1}{2}} \beta m_{\frac{1}{2}} \qquad m_0 = 0 \end{aligned} \tag{8}$$

this means that below $T_c = \frac{1}{2}$ there are four minima in the free energy surface corresponding to the four possible stable states

$$(\uparrow_1, \uparrow_2), (\uparrow_1, \downarrow_2), (\downarrow_1, \uparrow_2), (\downarrow_1, \downarrow_2) \tag{9}$$

where $\uparrow_1$ and $\uparrow_2$ represent the average spin directions in each of the two subsystems of $\frac{1}{2}N$ spins. These four states all have the same free energy. These results could, of course, have been obtained by a standard mean-field type of calculation on each of the two decoupled systems. We will now use the formalism we have developed for this system to study the coupled case. The effect of these couplings on the four stable states will be studied. In order to couple the two systems together in a translationally invariant way, such that the coupling between the two subsystems is weaker than those in the same subsystems, we can choose

$$J(r) = \frac{1}{N}\,(\,1\ \ \omega\ 1\ \ \omega\ \ldots\ 1\ \ \omega) \tag{10}$$

where $0 < \omega < 1$. With this choice for the interaction matrix, $\lambda_0$ and $\lambda_{\frac{1}{2}}$ are still the only non-zero eigenvalues, but now they take the values

$$\lambda_0 = \tfrac{1}{2}(1+\omega) \qquad \lambda_{\frac{1}{2}} = \tfrac{1}{2}(1-\omega). \tag{11}$$

As already mentioned, these generalized ferromagnetic models can be defined in the same way as the site-disorder spin glass and neural network models [4–7, 11] with deterministic choices for $\boldsymbol{\xi}_i$ (the quenched variables sitting on the sites). The model we are studying here can be expressed as

$$J_{ij} = \frac{1}{N}\sum_{\mu=1}^{2} \xi_i^{\mu}\xi_j^{\mu} \tag{12}$$

where $\xi_i^1 = \sqrt[+]{\frac{1}{2}(1+w)}$ for all $i$ and $\xi_i^2 = (-1)^i \sqrt[+]{\frac{1}{2}(1-w)}$. In general we can always choose $\xi_i^k = \sqrt{\lambda_k} V_k^i$ so that the interaction matrix can be defined by an equation with the form of (12). Therefore, the properties of the stable states of our ferromagnetic models are the same as those of the site-disorder spin glass models.

The effect of non-zero $\omega$ on the ferromagnetic states $(\uparrow_1, \uparrow_2)$ and $(\downarrow_1, \downarrow_2)$ is to increase their transition temperature to $T_c = \lambda_0 = \frac{1}{2}(1+\omega)$. The transition remains second order and the value of $m_0$ is given by the standard mean-field equation

$$m_0 = \tanh \beta\lambda_0 m_0. \tag{13}$$

The important new property of this coupled system is that the two non-ferromagnetic states $(\uparrow_1\downarrow_2)$ and $(\downarrow_1\uparrow_2)$ are still stable at low enough temperatures, and now appear discontinuously. At $T \leqslant \lambda_{\frac{1}{2}}$ a saddle point appears in the free energy surface (it bifurcates from the saddle point associated with the paramagnetic state) representing the two states $(\uparrow_1\downarrow_2)$ and $(\downarrow_1\uparrow_2)$, and it is only at a lower temperature, which we shall call $T_{\uparrow\downarrow}$, that it becomes a true minimum. This temperature can only be calculated numerically. A study of this Hessian matrix also shows that for $T_{\uparrow\downarrow} < T \leqslant \lambda_{\frac{1}{2}}$ the states are destabilized by fluctuations in the directions associated with ferromagnetic ordering. The non-ferromagnetic states are very similar in nature to the mixture states in neural networks (see reference [14]), in the sense that initially they occur as saddle points on the free energy surface, which stabilize at a lower temperature. The formation and bifurcation of minima and saddle points on the free energy surface has been studied in [4] for more complicated site-disorder spin glass problems.

We will now consider the case where all the interactions in the system have the same strength, but a spin in one subsystem does not interact with all the spins in the other subsystem. This can be done in a translationally invariant way by choosing the couplings between the two subsystems as

$$J(pn) = \frac{1}{N} \qquad \text{where} \quad n = 1, 3, 5, \ldots, N/p \ \ (p \ \text{odd}) \tag{14}$$

the systems being undefined when $p$ is not a factor of $N$. In this way we introduce for each spin $N/2p$ new interactions with spins in the other subsystem. The interaction matrix formed by coupling the two subsystems together in this way has the following non-zero eigenvalues

$$\begin{aligned} \lambda_0 &= \frac{1}{2} + \frac{1}{2p} \qquad \lambda_{n/2p} = +\frac{1}{2p} \qquad n = 2, 4, \ldots, 2p-2 \\ \lambda_{\frac{1}{2}} &= \frac{1}{2} - \frac{1}{2p} \qquad \lambda_{n/2p} = -\frac{1}{2p} \qquad n = 1, 3, \ldots, 2p-1 \ \ (n \neq p). \end{aligned} \tag{15}$$

In this paper we wish to study mean-field-type systems where the number of interactions per spin between the two subsystems scales with $N$. Choosing $p$ finite will give us systems of this type. The associated matrix has a finite number of non-zero eigenvalues with associated eigenvectors constructed from at most $2p$ distinct elements which are repeated $N/2p$ times. A system of this type satisfies the conditions for Curie–Weiss mean-field theory to be valid. We could also have coupled the system together with interactions associated with multiple choices of $p$ values and the conditions would also hold. Systems of this type, constructed from multiple choices of $p$, will be discussed later in this section.

In what follows, only minima of the free energy of the form $m_0 \neq 0, m_{k\neq 0} = 0$ and $m_{\frac{1}{2}} \neq 0, m_{k\neq\frac{1}{2}} = 0$ will be studied. Other stable states which depend on the choice of $p$ do exist but these states are best discussed after the next example. It should be noted that $\lambda_0$ and $\lambda_{\frac{1}{2}}$ are always the two largest eigenvalues of the matrix, so that condensation into states associated with their eigenvectors will always occur at a higher temperature than condensation into any of the other possible stable states of the system. We also see here for the first time the occurrence of negative eigenvalues. These do not play a role in the thermodynamics of the system, as we only consider condensation at positive temperatures which correspond to positive eigenvalues. It should also be noted in the context of equation (2) that our solutions satisfy

$$\tanh\left(\beta\sqrt{N} m_0 \lambda_0 V_0^i\right) = C(\beta) V_0^i \qquad \text{for all } i \tag{16}$$

where $C(\beta)$ is finite and only depends on $\beta$ and not $i$. Therefore, due to the orthogonality of the eigenvectors, $m_0 \neq 0$ and $m_{k\neq 0} = 0$ are valid solutions for this model, as was the case for the previous model. This equation is also true for the eigenvector $V_{\frac{1}{2}}$. This special property of the eigenvectors comes from the fact that the magnitudes of all their elements are the same. In general this will not be true. It is this special property of the eigenvectors which means that the behaviour of the stable states we are studying depends only on $p$ and not on the exact distribution of the other non-zero eigenvalues and their associated eigenvectors. If we associate $1/p$ with $\omega$ in the previous example, then the thermodynamics for the two models associated with the eigenvalues $\lambda_0$ and $\lambda_{\frac{1}{2}}$ is the same. The only difference between the two models is that for the partially connected case there is the possibility of more minima in the free energy surface.

We can now consider the most general case, where we choose an interaction matrix constructed from a finite set of finite $p$ values via the equation

$$J(pn) = \frac{1}{N} \qquad \text{where} \quad n = 1, 2, 3, \ldots, N/p \qquad \text{otherwise} \quad J(r) = \frac{w}{N} \tag{17}$$

which we shall denote $\{p_1, p_2, \ldots, p_s\}$ ($p = 2$ gives the previous example). Each choice of the $p_i$'s will have associated with it $p_i$ non-zero positive eigenvalues ($\lambda_{n/p}, n = 0, 1, \ldots, p-1$). $\lambda_0$ is always the largest eigenvalue corresponding to ferromagnetic ordering. Consequently for each $p_i$ we have this number of order parameters associated with the $p_i$ eigenvectors, which will characterize $2^{p_i}$ possible stable states. This is assuming, of course, that the temperature is low enough so that these states stabilize. The ferromagnetic state will always have the lowest free energy and will always be the first state to stabilize, the transition always being continuous. All other stable states will occur discontinuously, provided the system forms one cluster.

### 2.2. *Two weakly coupled systems at different critical temperatures*

An interesting extension of the above model is to break the translational invariance of the system and allow the two subsystems to have different interaction strengths defined by a matrix of the blocked form

$$\mathbf{J} = \frac{1}{N}\begin{pmatrix} J_1 & | & w \\ - & & - \\ w & | & J_2 \end{pmatrix} \tag{18}$$

where $J_1 > J_2 > w$. This model corresponds to weakly coupling together two systems with different critical temperatures. The non-zero eigenvalues of this matrix are given by

$$\lambda_\pm = \frac{1}{4}\left[J_1 + J_2 \pm \sqrt{(J_1 - J_2)^2 + 4w^2}\right] \tag{19}$$

with corresponding normalised eigenvectors,

$$V_\pm^i = \begin{cases} \sqrt{\dfrac{2}{N}}\,\dfrac{1}{[1 + ((2\lambda_\pm - J_1)/w)^2]^{\frac{1}{2}}} & \text{if } i \leqslant N/2 \\ \sqrt{\dfrac{2}{N}}\,\dfrac{2\lambda_\pm - J_1}{w[1 + ((2\lambda_\pm - J_1)/w)^2]^{\frac{1}{2}}} & \text{if } i > N/2. \end{cases} \tag{20}$$

The eigenvector corresponding to the largest eigenvalue $\lambda_+$ no longer corresponds to the magnetisation. An interesting property of this system is that at $T_c = \lambda_+$ the system condenses into a state where both the order parameters $m_+$ and $m_-$, corresponding to $V_\pm$, are non-zero. The order parameter $m_-$ is thus non-zero at a temperature above $\lambda_-$. For this state $m_+$ condenses out with a critical exponent of $\frac{1}{2}$ while $m_-$ condenses out with a critical exponent of $\frac{3}{2}$. The relative ratio of $m_+$ and $m_-$ is a function of the temperature. These properties, which differ from those studied in the previous example, are related to the fact that the solutions of the order parameter equations do not satisfy equation (16). This means that all the positive eigenvalues and associated eigenvectors can play a role in defining the properties of the stable states of the system. Some of this behaviour is reminiscent of the solutions of the TAP equations [15] for the SK model near $T_c$ [2] although it should be noted that our model is very different from the SK model since there is only a finite number of mean-field-type equations. Finally, we note that at low temperatures the system has a total of four stable states, like the model studied in the previous section, although the scenario for the formation of the minima in the free energy surface, associated with the non-ferromagnetic stable states, is different from the case $J_1 = J_2$. In the case $J_1 \neq J_2$ the minima associated with non-ferromagnetic states appear discontinuously at $T < \lambda_-$, and they do not develop from the unstable saddle points which appear at $T = \lambda_-$. For this model these saddle points do not develop into minima at any temperature.

## 3. Conclusion

This short paper has defined a class of long-range ferromagnetic models equivalent to site-disorder spin glass and neural network models. The most interesting property of these models is that they have many minima in the free energy surface, and thus have stable states which are non-ferromagnetic.

The work in this paper poses the question as to whether it is possible to define a long-range ferromagnetic system which can have 'true' spin glass behaviour similar to that of the SK spin glass [8, 2]. By 'true' spin glass behaviour we mean, as defined in [8], that at low temperature the number of minima in the free energy surface diverges in the thermodynamic limit corresponding to a divergent number of solutions to the TAP mean-field equations. As shown in reference [1], a necessary condition for an Ising spin system to have 'true' spin glass behaviour is that the rank of the interaction matrix must diverge in the thermodynamic limit. Thus the question is raised as to whether it is possible to choose a long-range generalized ferromagnetic system such that the rank of the interaction matrix is divergent in the thermodynamic limit (the weaker condition that all the eigenvalues are finite must of course be valid as well). We believe that the answer to this question is no, even though we do not have a general proof. When we add disorder into our ferromagnetic systems, while keeping the interactions ferromagnetic, the number of non-zero eigenvalues does increase (see examples in this paper) but their magnitude decreases (assuming we normalise the system such that the eigenvalue corresponding to the ferromagnetic state always has a finite value). This means that the rank of the interaction matrix cannot diverge in the thermodynamic limit, as the values of all but a finite set of the eigenvalues become vanishingly small. The long-range ferromagnetic bond disorder model studied in section 3 of [1] is perhaps the generalized ferromagnetic model closest to a spin glass. This model has interactions chosen to be 0 or $1/N$ with some probability $c$. The eigenvalue spectrum for this interaction matrix has $\lambda_0 = c$ (the eigenvalue corresponding to the ferromagnetic order parameter) while all the other eigenvalues are of order $\sqrt{1/N}$. Only the ferromagnetic state is stable at finite temperature ($T < T_c = c$), but at zero temperature all the modes associated with the other eigenvalues contribute to the partition function, and the system can have a diverging number of stable states in the thermodynamic limit.

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